SEMI-STABLE MINIMAL MODEL PROGRAM FOR VARIETIES WITH TRIVIAL CANONICAL DIVISOR

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Abstract. We give a sufficient condition for the termination of flips. Then we discuss a semi-stable minimal model program for varieties with (numerically) trivial canonical divisor as an application. We also treat a slight refinement of dlt blow-ups.

Contents

1. Introduction

In this paper, we give a sufficient condition for the termination of flips. For the precise statement, see Theorem 2.3. By using this criterion: Theorem 2.3, we prove the following theorem, which is a semi-stable minimal model program for varieties with trivial canonical divisor. It was inspired by Yoshinori Gongyo's paper [G2] and Daisuke Matsushita's seminar talk on May 21, 2010 in Kyoto.

Theorem 1.1 (Semi-stable minimal model program for varieties with trivial canonical divisor)**.** *Let $f : X \to Y$ be a proper surjective morphism from a smooth quasi-projective variety X to a smooth quasi-projective curve Y with connected fibers. Let $P \in Y$ be a point. Assume that f^*P is a reduced simple normal crossing divisor on X and*

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f is smooth over $Y \setminus P$. We further assume that $K_{f^{-1}Q} \sim 0$ for every $Q \in Y \setminus P$. Then there exists a sequence of flips and divisorial contractions

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_k \dashrightarrow \cdots \dashrightarrow X_m$$

over Y such that $K_{X_m} \sim_Y 0$. We note that X_m has only $\mathbb{Q}$-factorial terminal singularities. Moreover, the special fiber $S = f_m^{-1}P = f_m^*P$ of $f_m : X_m \to Y$ is Gorenstein, semi divisorial log terminal, and $K_S \sim 0$.

For the definition of *semi divisorial log terminal*, see [F1, Definition 1.1]. For the proof of the termination of 4-dimensional semi-stable log flips, see [F2]. Theorem 1.1 can be applied to semi-stable degenerations of Abelian varieties, Calabi-Yau varieties, and so on. From the minimal model theoretic viewpoint, the following theorem is a natural formulation of uniruled degenerations of varieties with numerically trivial canonical divisor (cf. [T, Theorem 1.1]).

Theorem 1.2 (Semi-stable minimal model program for varieties with numerically trivial canonical divisor)**.** *Let $f : X \to Y$ be a proper surjective morphism from a smooth quasi-projective variety X to a smooth quasi-projective curve Y with connected fibers. Let $P \in Y$ be a point. Assume that f^*P is a reduced simple normal crossing divisor on X and f is smooth over $Y \setminus P$. We further assume that $K_{f^{-1}Q} \equiv 0$, equivalently, $K_{f^{-1}Q} \sim_{\mathbb{Q}} 0$, for every $Q \in Y \setminus P$. Then there exists a sequence of flips and divisorial contractions*

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_k \dashrightarrow \cdots \dashrightarrow X_m$$

over Y such that $K_{X_m} \sim_{\mathbb{Q},Y} 0$. We note that X_m has only $\mathbb{Q}$-factorial terminal singularities. Moreover, the special fiber $S = f_m^{-1}P = f_m^*P$ of $f_m : X_m \to Y$ is semi divisorial log terminal and $K_S \sim_{\mathbb{Q}} 0$. Therefore, if S is reducible, then every irreducible component of S is uniruled. If S is irreducible, then S is uniruled if and only if S is not canonical.

In this paper, we prove Theorem 1.1 and Theorem 1.2 as applications of the following theorem.

Theorem 1.3. *Let (X, Δ) be a $\mathbb{Q}$-factorial quasi-projective divisorial log terminal pair and let $f : X \to Y$ be a proper surjective morphism onto a smooth quasi-projective curve Y with connected fibers. Assume that $(K_X + \Delta)|_F \sim_{\mathbb{Q}} 0$ for a general fiber F of f. Then there exists a sequence of flips and divisorial contractions*

$$\begin{aligned}(X, \Delta) = (X_0, \Delta_0) &\dashrightarrow (X_1, \Delta_1) \dashrightarrow \cdots \\ &\dashrightarrow (X_k, \Delta_k) \dashrightarrow \cdots \dashrightarrow (X_m, \Delta_m)\end{aligned}$$

over Y such that $K_{X_m}+\Delta_m \sim_{\mathbb{Q},Y} 0$ where Δ_k is the pushforward of Δ on X_k for every k.

Remark 1.4. It is known that $(K_X+\Delta)|_F \sim_{\mathbb{Q}} 0$ if and only if $(K_X+\Delta)|_F \equiv 0$. See, for example, [CKP, Theorem 1] and [G2, Theorem 1.2].

We can also prove the following theorem as an application of Theorem 1.3. We recommend the reader to compare it with Kodaira's classification of elliptic fibrations (cf. [BPV, V. Examples]).

Theorem 1.5 (cf. [T, Theorem 1.1])**.** *Let $f: X \to Y$ be a proper surjective morphism from a smooth quasi-projective variety X to a smooth quasi-projective curve Y with connected fibers. Let $P \in Y$ be a point. Assume that $\mathrm{Supp} f^*P$ is a simple normal crossing divisor on X and f is smooth over $Y \setminus P$. We further assume that $K_{f^{-1}Q} \equiv 0$, equivalently, $K_{f^{-1}Q} \sim_{\mathbb{Q}} 0$, for every $Q \in Y \setminus P$. Then there exists a sequence of flips and divisorial contractions*

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_k \dashrightarrow \cdots \dashrightarrow X_m$$

over Y such that X_m has only $\mathbb{Q}$-factorial terminal singularities and $K_{X_m} \sim_{\mathbb{Q},Y} 0$. Let $S = \mathrm{Supp} f_m^*P$ be the special fiber of $f_m : X_m \to Y$. If S is reducible, then every irreducible component of S is uniruled. If S is irreducible, then S is normal and has only canonical singularities if and only if S is not uniruled. We note that $K_S \sim_{\mathbb{Q}} 0$ when S is irreducible and has only canonical singularities.

By combining Theorem 1.3 with [L, Proposition 2.7], we obtain the following result.

Corollary 1.6. *Let $f: X \to Y$ be a projective surjective morphism from a smooth quasi-projective variety X onto a smooth quasi-projective curve Y with connected fibers. Assume that the general fiber F of f has a good minimal model and $\kappa(F) = 0$, where $\kappa(F)$ is the Kodaira dimension of F. Then there exists a sequence of flips and divisorial contractions*

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_k \dashrightarrow \cdots \dashrightarrow X_m$$

over Y such that $K_{X_m} \sim_{\mathbb{Q},Y} 0$.

Remark 1.7. By [D, Corollaire 3.4], F has a good minimal model with $\kappa(F) = 0$ if and only if $\kappa_\sigma(F) = 0$, where $\kappa_\sigma(F)$ is the numerical Kodaira dimension in the sense of Nakayama. See also [G2, Theorem 1.2].

Finally, in Section 4, we treat a slight refinement of *dlt blow-ups* (cf. Theorem 4.1) as an application of our criterion for the termination of flips: Theorem 2.3, which generalizes [Ko, 17.10 Theorem] and

[BCHM, Corollary 1.4.3]. We will use Theorem 4.1 in the proofs of Theorem 1.2 and Theorem 1.5.

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Notation. Let X be a normal variety and let $D = \sum_i a_i D_i$ be an $\mathbb{R}$-divisor on X, where D_i is a prime divisor and $a_i \in \mathbb{R}$ for every i and $D_i \neq D_j$ for every $i \neq j$. In this case, D is called $\mathbb{R}$-*boudnary* if and only if $0 \leq a_i \leq 1$ for every i.

Let $f : X \to Y$ be a proper morphism of normal algebraic varieties. Two $\mathbb{Q}$-divisors D_1 and D_2 on X are $\mathbb{Q}$-*linearly equivalent over* Y, denoted by $D_1 \sim_{\mathbb{Q},Y} D_2$, if their difference is a $\mathbb{Q}$-linear combination of principal divisors and a $\mathbb{Q}$-Cartier divisor pulled back from Y.

Let X be a normal variety and let Δ be an $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let E be a divisor *over* X. Then the *discrepancy* coefficient of E with respect to (X, Δ) is denoted by $a(E, X, \Delta)$.

We work over $\mathbb{C}$, the complex number field, throughout this paper. We freely use the standard terminology on the log minimal model program in [BCHM] and [KM].

2. Easy termination lemma

In this section, we give a sufficient condition for the termination of flips. First, let us recall the definitions of *movable divisors* and the *movable cone*.

Definition 2.1 (Movable divisors and movable cone)**.** Let $f : X \to Y$ be a projective morphism of normal algebraic varieties. A Cartier divisor D on X is called f-*movable* if $f_*\mathcal{O}_X(D) \neq 0$ and if the cokernel of the natural homomorphism $f^*f_*\mathcal{O}_X(D) \to \mathcal{O}_X(D)$ has a support of codimension ≥ 2.

Let M be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. Then M is called f-*movable* if and only if $M = \sum_i a_i D_i$ where a_i is a positive real number and D_i is an f-movable Cartier divisor for every i.

We define $\overline{\mathrm{Mov}}(X/Y)$ as the closed convex cone in $N^1(X/Y)$, which is called the *movable cone* of $f : X \to Y$, generated by the classes of f-movable Cartier divisors.

Let us recall the minimal model program with scaling (cf. [BCHM, 3.10], [B, Definition 3.2], and [F4, Theorem 18.9]).

2.2 (Minimal model program with scaling)**.** Let (X, Δ) be a $\mathbb{Q}$-factorial dlt pair such that Δ is an $\mathbb{R}$-divisor and let $f : X \to Y$ be a projective surjective morphism between quasi-projective varieties. Let H be an effective $\mathbb{R}$-divisor on X such that $(X, \Delta + H)$ is divisorial log terminal, $K_X + \Delta + H$ is f-nef, and the relative augmented base locus $\mathbf{B}_+(H/Y)$ (cf. [BCHM, Definition 3.5.1]) contains no lc centers of (X, Δ). We run the $(K_X + \Delta)$-minimal model program with scaling of H over Y. We obtain a sequence of divisorial contractions and flips

$$(X, \Delta) = (X_0, \Delta_0) \dashrightarrow (X_1, \Delta_1) \dashrightarrow \cdots \dashrightarrow (X_k, \Delta_k) \dashrightarrow \cdots$$

over Y. We note that

$$\lambda_i = \inf\{t \in \mathbb{R} \,|\, K_{X_i} + \Delta_i + tH_i \text{ is nef over } Y\},$$

where H_i (resp. Δ_i) is the pushforward of H (resp. Δ) on X_i for every i. By the definition, $0 \le \lambda_i \le 1$ and $\lambda_i \in \mathbb{R}$ for every i and

$$\lambda_0 \ge \lambda_1 \ge \cdots \ge \lambda_k \ge \cdots .$$

We also note that the relative augmented base locus $\mathbf{B}_+(H_i/Y)$ contains no lc centers of (X_i, Δ_i) for every i (cf. [BCHM, Lemma 3.10.11]).

The following theorem is the main result of this section.

Theorem 2.3 (Easy termination lemma)**.** *Under the same notation as in* 2.2*, we assume that H is big over Y, every step of the $(K_X + \Delta)$-minimal model program is a flip, and $K_X + \Delta \notin \overline{\mathrm{Mov}}(X/Y)$. Then it terminates after finitely many steps.*

Proof. We assume that the sequence does not terminate. First we assume that

$$\lambda = \lim_{i \to \infty} \lambda_i > 0.$$

In this case, the sequence of flips we consider is a sequence of $(K_X + \Delta + \frac{1}{2}\lambda H)$-flips. We note that there exists an effective $\mathbb{R}$-divisor B on X such that $\Delta + \frac{1}{2}\lambda H \sim_{\mathbb{R}} B$, (X, B) is klt, $K_X + B + (1 - \frac{1}{2}\lambda)H$ is f-nef, $(X, B + (1 - \frac{1}{2}\lambda)H)$ is klt, and B is big over Y (cf. [BCHM, Lemma 3.7.3] and [G2, Lemma 5.1]). Therefore there are no infinite sequences of flips by [BCHM, Corollary 1.4.2]. It is a contradiction. Thus we can assume that $\lambda = 0$. Under the assumption that $\lambda = 0$, we will show that $K_X + \Delta \in \overline{\mathrm{Mov}}(X/Y)$. Let G_i be a relative ample $\mathbb{Q}$-divisor on X_i such that $G_{iX} \to 0$ in $N^1(X/Y)$ for $i \to \infty$ where G_{iX} is the strict transform of G_i on X. We note that $K_{X_i} + \Delta_i + \lambda_i H_i + G_i$ is ample over Y for every i. Therefore the strict transform $K_X + \Delta + \lambda_i H + G_{iX}$

is movable on X for every i. Thus $K_X + \Delta$ is a limit of movable $\mathbb{R}$-divisors in $N^1(X/Y)$. So $K_X + \Delta \in \overline{\mathrm{Mov}}(X/Y)$. It is a contradiction. Therefore the sequence of flips terminates after finitely many steps. □

3. Proofs

In this section, we will prove various results stated in Section 1 as applications of Theorem 2.3.

Proof of Theorem 1.3. Before we run the minimal model program with scaling, we note the following easy observation.

Step 1 (cf. [FM, Proposition 4.2])**.** Let m be a positive integer such that $m(K_X + \Delta)$ is Cartier and $m(K_X + \Delta)|_F \sim 0$ where F is the generic fiber of f. Then we have a natural injection

$$0 \to f^* f_* \mathcal{O}_X(m(K_X + \Delta)) \to \mathcal{O}_X(m(K_X + \Delta))$$

because $f_*\mathcal{O}_X(m(K_X + \Delta))$ is torsion-free and Y is a smooth curve. Therefore, there is a $\mathbb{Q}$-divisor D on Y and an effective $\mathbb{Q}$-divisor B on X such that B is vertical with respect to f,

$$K_X + \Delta \sim_{\mathbb{Q}} f^*D + B,$$

and $\mathrm{Supp} B$ does not contain any fibers of f. We note that $K_X + \Delta$ is f-nef if and only if $B = 0$, equivalently, $K_X + \Delta \sim_{\mathbb{Q},Y} 0$ (cf. [BPV, III. (8.2) Lemma]).

Step 2. We take an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor H on X such that H is big, $(X, \Delta + H)$ is dlt, $K_X + \Delta + H$ is nef over Y, and $\mathbf{B}_+(H/Y)$ contains no lc centers of (X, Δ). We run the $(K_X + \Delta)$-minimal model program with scaling of H over Y as in 2.2. Since divisorial contractions can occur only finitely many times, we can assume that every step is a flip. Since $B \not\sim_{\mathbb{Q},Y} 0$, we can find an irreducible component E of $\mathrm{Supp} B$ such that

$$B \cdot A^{n-2} \cdot E < 0.$$

where $n = \dim X$ and A is an f-ample Cartier divisor on X. This is essentially Zariski's lemma (cf. [BPV, III. (8.2) Lemma]). Thus

$$(K_X + \Delta) \cdot A^{n-2} \cdot E < 0.$$

Assume that $K_X + \Delta \in \overline{\mathrm{Mov}}(X/Y)$. Then

$$(K_X + \Delta) \cdot A^{n-2} \cdot E \geq 0.$$

Therefore, $K_X + \Delta \notin \overline{\mathrm{Mov}}(X/Y)$. Thus the $(K_X + \Delta)$-minimal model program terminates by Theorem 2.3.

Step 3. On the output X_m of the minimal model program, $K_{X_m} + \Delta_m \sim_{\mathbb{Q},Y} B_m$ where B_m is the pushforward of B on X_m. Since B_m is nef over Y, $B_m \sim_{\mathbb{Q},Y} 0$ (cf. [BPV, III. (8.2) Lemma]). Therefore, $K_{X_m} + \Delta_m \sim_{\mathbb{Q},Y} 0$.

We complete the proof of Theorem 1.3. $\square$

Remark 3.1. Let $f : (X, \Delta) \to Y$ be a projective dlt morphism from a $\mathbb{Q}$-factorial dlt pair (X, Δ) (cf. [KM, Definition 7.1]). Assume that $K_X + \Delta$ is f-nef over a non-empty Zariski open set $U \subset Y$. Then the special termination (see, for example, [F3, Theorem 4.2.1]) implies that any sequence of flips in the $(K_X + \Delta)$-minimal model program over Y must terminate. We note that the special termination has been proved only in dimension ≤ 4 (see, for example, [F3, Theorem 4.2.1]).

Let us prove Theorems 1.1, 1.2, 1.5, and Corollary 1.6.

Proof of Theorem 1.1. By the assumptions, $f : X \to Y$ is a dlt morphism (cf. [KM, Definition 7.1]). By applying Theorem 1.3, we obtain a relative minimal model $f_m : X_m \to Y$ of $f : X \to Y$. We see that $f_m : X_m \to Y$ is automatically a dlt morphism. We note that X_m is $\mathbb{Q}$-factorial and has only terminal singularities. By adjunction,

$$(K_{X_m} + S)|_S = K_S$$

and S is semi divisorial log terminal because (X_m, S) is dlt (cf. [F1, Remark 1.2 (3)]). By considering the following natural injection

$$0 \to f^* f_* \mathcal{O}_{X_m}(K_{X_m}) \to \mathcal{O}_{X_m}(K_{X_m}),$$

which is also surjective outside the special fiber S, as in Step 1 in the proof of Theorem 1.3, we obtain $K_{X_m} \sim 0$ because K_{X_m} is nef over Y. In particular, $K_S \sim 0$ by adjunction. $\square$

Proof of Theorem 1.2. The proof of Theorem 1.1 works in this setting. If S is reducible, semi divisorial log terminal, and $K_S \sim_{\mathbb{Q}} 0$, then we will show that every irreducible component of S is uniruled. Let S_0 be an irreducible component of S. Then $K_{S_0} + \Theta \sim_{\mathbb{Q}} 0$ with an effective $\mathbb{Q}$-divisor $\Theta \neq 0$ because S is connected. Therefore, S_0 is uniruled by [MM, Corollary 2]. From now on, we assume that S is irreducible. If S has only canonical singularities, then S is not uniruled because $K_S \sim_{\mathbb{Q}} 0$. If S is not canonical, then we take a dlt blow-up (cf. Theorem 4.1) and obtain a birational morphism $\varphi : T \to S$ from a normal projective variety T such that $K_T = \varphi^* K_S - E$ where E is effective and $E \neq 0$. Therefore, $K_T \sim_{\mathbb{Q}} -E \neq 0$. Thus T is uniruled by [MM, Corollary 2]. It implies that S is uniruled. $\square$

Proof of Theorem 1.5. The former part follows from Theorem 1.3. We will check the latter part. We assume that S is reducible. Let E be any irreducible component of S, and let ε be a sufficiently small positive rational number. Apply Theorem 1.3 to $(X, \varepsilon E)$ over Y. Then it is easy to see that the divisor E must be contracted in this minimal model program. Therefore E is uniruled by [KMM, Proposition 5-1-8]. From now on, we assume that S is irreducible. It is sufficient to see that S is uniruled when S is not canonical. First we assume that S is normal. Then we take a dlt blow-up $\varphi : T \to S$ (cf. Theorem 4.1). We can write $K_T = \varphi^* K_S - E$ such that $E \neq 0$ is effective. Therefore, T is uniruled by [MM, Corollary 2] because $K_T \sim_{\mathbb{Q}} -E \neq 0$. Thus S is uniruled. Next we assume that S is not normal. Let $\nu : S^\nu \to S$ be the normalization. Then

$$K_{S^\nu} + \Theta = \nu^* K_S \sim_{\mathbb{Q}} 0$$

such that Θ is effective and $\Theta \neq 0$. We note that S is Cohen–Macaulay since X is Cohen–Macaulay and S is $\mathbb{Q}$-Cartier (cf. [KM, Corollary 5.25]). Therefore, S^ν is uniruled by [MM, Corollary 2]. Thus S is uniruled. Anyway, S is not uniruled if and only if S has only canonical singularities. □

Proof of Corollary 1.6. Let H be a general effective f-big divisor on X such that $K_X + H$ is f-nef and (X, H) is dlt. We run the minimal model program with scaling of H over Y. Then, by [L, Proposition 2.7], we can assume that the general fiber of $f : X \to Y$ is a good minimal model. By Theorem 1.3, this minimal model program terminates after finitely many steps. □

4. Dlt blow-ups

In this section, we will give a slight refinement of [Ko, 17.10 Theorem] and [BCHM, Corollary 1.4.3] as an application of Theorem 2.3. See also [F4, §10].

Theorem 4.1 (Dlt blow-ups)**.** *Let X be a normal quasi-projective variety and let Δ be an $\mathbb{R}$-boundary divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $f : W \to X$ be a resolution such that* $\mathrm{Exc}(f) \cup \mathrm{Supp} f_*^{-1}\Delta$ *is a simple normal crossing divisor on W where* $\mathrm{Exc}(f)$ *is the exceptional locus of f. Let $\mathcal{E}$ be a subset of the f-exceptional divisors $\{E_i\}$ with the following properties*:

- *If $a(E_i, X, \Delta) \leq -1$, then $E_i \in \mathcal{E}$.*
- *If $E_i \in \mathcal{E}$, then $a(E_i, X, \Delta) \leq 0$.*

Then there is a factorization

$$f : W \overset{h}{\dashrightarrow} Z \xrightarrow{g} X$$

with the following properties:

(a) *h is a local isomorphism at every generic point of $E_i \in \mathcal{E}$,*
(b) *h contracts every exceptional divisor not in $\mathcal{E}$,*
(c) *we have*

$$\begin{aligned} &h_*(K_W + f_*^{-1}\Delta + \sum_{a_i \geq -1} -a_iE_i + \sum_{a_i < -1} E_i) \\ &= K_Z + g_*^{-1}\Delta + \sum_{E_i \in \mathcal{E},\ a_i \geq -1} -a_ih_*E_i + \sum_{a_i<-1} h_*E_i \\ &= g^*(K_X + \Delta) + \sum_{a_i<-1} (a_i + 1)h_*E_i, \end{aligned}$$

where $a_i = a(E_i, X, \Delta)$, and
(d) *the pair*

$$(Z, g_*^{-1}\Delta + \sum_{E_i \in \mathcal{E},\ a_i \geq -1} -a_ih_*E_i + \sum_{a_i<-1} h_*E_i)$$

is a $\mathbb{Q}$-factorial dlt pair.

In particular, if (X, Δ) is log canonical, then

$$(Z, g_*^{-1}\Delta + \sum_{E_i \in \mathcal{E}, a_i \geq -1} -a_ih_*E_i)$$

is dlt and

$$K_Z + g_*^{-1}\Delta + \sum_{E_i \in \mathcal{E}, a_i \geq -1} -a_ih_*E_i = g^*(K_X + \Delta).$$

Proof. For a small $\varepsilon > 0$, we put

$$d(E_i) = \begin{cases} 1 & \text{if } \ a(E_i, X, \Delta) < -1 \\ -a(E_i, X, \Delta) & \text{if } \ E_i \in \mathcal{E}, a(E_i, X, \Delta) \geq -1 \\ \max\{-a(E_i, X, \Delta) + \varepsilon, 0\} & \text{if } \ E_i \notin \mathcal{E}. \end{cases}$$

We take a general effective Cartier divisor H on Z such that $(W, f_*^{-1}\Delta + \sum d(E_i)E_i + H)$ is dlt and that $K_W + f_*^{-1}\Delta + \sum d(E_i)E_i + H$ is f-nef. We run the $(K_W + f_*^{-1}\Delta + \sum d(E_i)E_i)$-minimal model program with scaling of H over X. We note that

$$\begin{aligned} &K_W + f_*^{-1}\Delta + \sum d(E_i)E_i \\ &= f^*(K_X + \Delta) + \sum_{E_i \notin \mathcal{E}} (d(E_i) + a_i)E_i + \sum_{a_i<-1} (1 + a_i)E_i. \end{aligned}$$

Since divisorial contractions can occur finitely many times, we can assume that every step of the minimal model program is a flip. We put

$$E=\sum_{E_i\not\in\mathcal{E}}(d(E_i)+a_i)E_i+\sum_{a_i<-1}(1+a_i)E_i.$$

Then E is exceptional over X. We assume that $\sum_{E_i\not\in\mathcal{E}}(d(E_i)+a_i)E_i\neq 0$. Then $E\not\in\overline{\mathrm{Mov}}(W/X)$ by Lemma 4.2 below. Therefore, any sequence of flips terminates after finitely many steps by Theorem 2.3. However, E can not become nef over X by flips since $-E$ is not effective. It is a contradiction. Therefore, $\sum_{E_i\not\in\mathcal{E}}(d(E_i)+a_i)E_i=0$. It completes the proof. □

The lemma below is a variant of the well-known negativity lemma.

Lemma 4.2. *Let $f:X\to Y$ be a birational morphism from a normal $\mathbb{Q}$-factorial algebraic variety X. Let E be an $\mathbb{R}$-divisor on X such that* $\mathrm{Supp}E$ *is f-exceptional and $E\in\overline{\mathrm{Mov}}(X/Y)$. Then $-E$ is effective.*

Proof. We write $E=E_+-E_-$ such that E_+ and E_- have no common irreducible components and that $E_+\geq 0$ and $E_-\geq 0$. We assume that $E_+\neq 0$. Let A (resp. H) be an ample Cartier divisor on Y (resp. X). Then we can find an irreducible component E_0 of E_+ such that

$$E_0\cdot (f^*A)^k\cdot H^{n-k-2}\cdot E<0$$

where $\dim X=n$ and $\mathrm{codim}_Y f(E_+)=k$. On the other hand,

$$E_0\cdot (f^*A)^k\cdot H^{n-k-2}\cdot E\geq 0$$

if $E\in\overline{\mathrm{Mov}}(X/Y)$. It is a contradiction. Therefore, $-E$ is effective. □

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